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A Multiaxial Differential Model of Flow in Orthotropic Polycrystalline Ice

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ABSTRACT

A multiaxial differential model is proposed for pure flow in orthotropic polycrystalline ice. The derivation of the constitutive equations is based on thermodynamics with internal state variables. The model equations consist of the equations-of-state and evolution equations for the internal variables and a nonelastic deformation variable. The internal state of the material is described in terms of a scalar and a second rank tensor, which represent isotropic and kinematic hardening in the material, respectively. The nonelastic deformation-rate tensor is additively decomposed into transient and steady-state components. The orthotropic texture of ice during incompressible flow is characterized by five material parameters which define appropriate measures of the thermodynamic forces and deformations. Conventionally used mechanical tests under constant-stress creep and constant strain-rate loading are sufficient to determine these parameters.

1. INTRODUCTION

The mechanical behavior of ice is rate and temperature dependent since it is generally present in nature at very high homologous temperatures. Solutions to complex problems in applied ice mechanics, involving multiaxial states of stress and which recognize this rate-dependence, are based on a generalization of Glen's (1955) power-law model of steady-flow that assumes polycrystalline ice to be isotropic and incompressible. Such a generalization has been proposed for glacier-flow problems by Palmer (1967) and applied to ice-indentation problems by Ponter et al. (1983).

Ice, however, is not isotropic in many engineering problems. The most prevalent polymorphic form of ice, Ice-Ih, has a hexagonal crystal structure (Hobbs, 1974). The oxygen atoms are all concentrated along basal planes perpendicular to the principal hexagonal axis, called the c-axis. Single crystals of this type of ice are cross-anisotropic or transversely-isotropic because of hexagonal symmetry about the c-axis. The c-axes of different crystals in polycrystalline ice can exhibit various degrees of alignment, thereby imparting a definite crystallographic texture or fabric to the solid.

The individual crystals in polycrystalline ice may be granular, columnar, or in one of many other, generally, less common forms. Granular ice is typically isotropic. On the other hand, the texture of columnar ice may be isotropic, transversely-isotropic, or more generally anisotropic. According to the classification of fresh-water polycrystalline ice by Michel and Ramseier (1971), transverse-isotropy is exhibited by S-2 ice, which is commonly encountered in ice engineering problems, while more general anisotropy is exhibited by S-3 ice. The power-law model of steady-flow has been generalized for

orthotropic and incompressible material behavior and applied to ice-indentation problems by Shyam Sunder et al. (1987).

These multiaxial models of flow, however, neglect the transient component of deformation which can be important in a broad range of ice mechanics problems (Gold, 1977; Sinha et al., 1987). In particular, the transition from ductile to brittle behavior in polycrystalline ice is generally initiated during transient flow. For example, appreciable transient flow, comparable in magnitude to the elastic deformation, precedes failure in ice under uniaxial tension (Schulson et al., 1984; Schulson, 1987). Second, transient-flow is essential for predicting "first-crack occurrence" or "crack nucleation" during uniaxial compressive loading (Sinha, 1982; Shyam Sunder and Ting, 1985). Finally, at loading rates of engineering interest (viz., 10^{-6} to 10^{-1} s^{-1}), transient-flow may be followed by strain-softening without a region dominated by steady-flow because of damage due to distributed cracking. Similarly, primary creep may be followed by tertiary (accelerating) creep without appreciable secondary (steady) creep under constant-stress loading. However, existing models based on damage mechanics are incapable of characterizing the ductile-to-brittle transition in ice since they ignore the contribution of transient deformation to the total deformation response of the material. This is illustrated by Sjölind's (1987) formulation which does not account for the contribution of transient deformation to the thermodynamic free energy and the dissipation potential function.

A uniaxial differential model which explicitly accounts for transient-flow in polycrystalline ice has been presented by Shyam Sunder and Wu (1988). The model satisfies the dimensional requirements identified by Ashby and Duval (1985). Correspondence between constant-stress creep and constant strain-rate response, observed by Mellor and Cole (1982, 1983), has been established. The uniaxial modle contains a total of six parameters and follows experimental data on the creep of fresh-water polycrystalline ice obtained by Jacka (1984), Sinha (1978), and Brill and Camp (reproduced in Sinha, 1979).

This paper presents a multiaxial generalization of the uniaxial differential model for flow in orthotropic polycrystalline ice. The derivation of the constitutive equations is based on thermodynamics with internal state variables (Coleman and Gurtin, 1967). The model equations consist of the equations-of-state and evolution equations for the internal variables and the nonelastic deformation variable. The internal state of the material is described by two internal variables: a scalar and a second rank symmetric tensor which represent, in an averaging sense, structural changes occurring on the microscale. Such approximate description of the internal state has been widely adopted; an example being the constituive model for the hot-working of metals developed by Brown et al. (1987). The total deformation-rate tensor is additively decomposed into elastic and nonelastic components. The nonelastic component is further additively decomposed into steady-state and transient components.

The equations-of-state, relating the state variables and their conjugate thermodynamic forces, are derived from an admissible thermodynamic (Helmholtz) free energy potential. The Helmholtz free energy consists of contributions from : (a) the elastic-deformation tensor; (b) an anelastic-deformation tensor associated with the elastic back-stresses generated during kinematic hardening; and (c) a scalar measure

associated with drag-stresses generated during isotropic hardening.

The dissipation for steady-flow is represented by an independent potential function from which the corresponding flux quantity is obtained by imposing normality. The dissipation for transient-flow is characterized by a complex potential function that seeks to model the coupling between isotropic and kinematic hardening. It is postulated that the transient deformation-rate and the fluxes for the internal variables are derivable from the complex potential by imposing normality with respect to the associated thermodynamic forces.

Material or texture anisotropy effects during transient- and steady-flow are modeled orthotropically by defining equivalent values for the thermodynamic force tensors in a manner analogous to the Hill-criterion of plastic yield. The five parameters which are used to describe the orthotropic texture of polycrystalline ice can be estimated from constant-stress creep or constant strain-rate tests under steady-flow conditions. Conventional uniaxial compression testing is sufficient to determine these parameters.

2. THERMODYNAMIC FORMULATION

Thermodynamic Laws and the Clausius-Duhem Inequality.-- In Coleman annd Gurtin's (1967) approach, deformations are viewed as processes rather than states of equilibrium or near-equilibrium. A thermodynamical process is specified directly by constitutive functions defined on material points of the continuum. The assumption is that a set of physically significant thermo-mechanical quantities can be defined on any particle in a material body. The particle may be visualized as a microscopic cell, which is itself a continuum in the microscopic description. Since it may consist of highly heterogeneous entities, e.g., an aggregate of grains, it must be large enough to represent the average behavior of the aggregate, and yet small enough compared to the dimensions of the continuum (e.g., solid body) to be associated with the material point. To these cells is applied the homogenization process whereby the locally heterogeneous medium is replaced by a homogeneous medium. The thermo-mechanical state of each material point can then be described by variables representing the average behavior of the cell. These include independent (observable) state variables like the temperature, the temperature gradient and the strain tensor, and internal variables which generally characterize, in an approximate sense, the various history-dependent mechanisms of structural changes.

Only small strains and rotations are considered in the present model. If only isothermal processes are considered, and if thermal dissipation is neglected, the response of the material at any instant is governed by (i) the independent variable, i.e., the total strain tensor, $\underline{\varepsilon}$; (ii) the nonelastic strain tensor, $\underline{\varepsilon}_n$; and (iii) the internal variables, i.e., the second rank symmetric tensor associated with kinematic hardening, $\underline{\alpha}$, and the scalar measure associated with isotropic hardening, p. However, the nonelastic strain tensor represents a macroscopic deformation due to internal structural changes and is not regarded as a state variable. It is determined from a constitutive function which is dependent on the internal state variables. Given this and the additive decomposition of the total strain-rate into elastic and nonelastic components, the total strain tensor may be replaced by the elastic strain tensor $\underline{\varepsilon}_e$ as a state variable. Thus, the state variables are given by the list $\{\underline{\varepsilon}_e, \underline{\alpha}, p\}$, or alternatively by the conjugate variables $\{\underline{\sigma}, \underline{R},$

B'V}, where $\underline{\alpha}$, $\underline{R}$, and B'V denote the stress tensor, the back-stress tensor and the scalar drag-stress, respectively. The Helmholtz free energy and the dissipation potential, from which all the model equations are derived, are functions of these state variables.

The first law of thermodynamics is an energy conservation equatin. For small strains and rotations the local form of this law balances the rate of internal energy per unit volume $\rho\dot{U}$, the stress power $\underline{\sigma}:\dot{\underline{\varepsilon}}$, the divergence of the heat-flux vector div $\underline{q}$, and the heat supply per unit volume ρr as follows:

$$\rho\dot{U} = \underline{\sigma} : \dot{\underline{\varepsilon}} + \rho r - \text{div}\,\underline{q} \tag{1}$$

where ρ is the mass density, and $\dot{\underline{\varepsilon}}$is the strain-rate tensor. The symbol ':' denotes tensor contraction, i.e., $\underline{A} : \underline{B} = A_{ij}B_{ij}$, while the superposed dot denotes the material time derivative. The second law, in the form of the Clausius-Duhem inequality, asserts that the rate of entropy production per unit volume $\rho\dot{\gamma}$ is non-negative:

$$\rho\dot{\gamma} = \rho\dot{\eta} - \rho r/T + \text{div}(\underline{q}/T) \geq 0 \tag{2}$$

where $\rho\dot{\eta}$ is the rate of entropy increase per unit volume and T is the absolute temperature. By a partial Legendre transform, the dependence of the internal energy on entropy is replaced by the dependence on absolute temperature. This yields the Helmholtz free energy (per unit mass) $A = U - T\eta$. Applying this transformation to Eq. (1) and substituting the result in inequality (2) gives rise to the alternative form of the Clausius-Duhem inequality (called the reduced dissipation inequality in the literature):

$$-\rho(\dot{A} + \eta\dot{T}) + \underline{\sigma} : \dot{\underline{\varepsilon}} - \underline{q} \bullet \text{grad}(\log T) = \rho T\dot{\gamma} \geq 0 \tag{3}$$

where the single dot between two vectors denotes the vector dot product. For an isothermal process, $\dot{T}$= 0. Furthermore, if the thermal gradient is negligible, i.e., the temperature field is approximately homogeneous within the microscopic cell defining a particle in the material, inequality (3) can be re-written in the following form:

$$-\rho\dot{A} + \underline{\sigma} : \dot{\underline{\varepsilon}} \geq 0 \tag{4}$$

Assuming that $A \equiv A(\underline{\varepsilon}_e, \underline{\alpha}, p)$ is a smooth function of the state variables, the following equation must be valid for admissible thermodynamic processes:

$$\dot{A} = \frac{\partial A}{\partial \underline{\varepsilon}_e} : \dot{\underline{\varepsilon}}_e + \frac{\partial A}{\partial \underline{\alpha}} : \dot{\underline{\alpha}} + \frac{\partial A}{\partial p}\dot{p} \tag{5}$$

Noting that $\dot{\underline{\varepsilon}} = \dot{\underline{\varepsilon}}_e + \dot{\underline{\varepsilon}}_n = \dot{\underline{\varepsilon}}_e + \dot{\underline{\varepsilon}}_t + \dot{\underline{\varepsilon}}_v$, where the last two quantities denote respectively the transient and steady-state components of the nonelastic strain-rate, inequality (4) and Eq. (5) can be combined to yield:

$$[\underline{\sigma} - \rho \frac{\partial A}{\partial \underline{\varepsilon}_e}] : \dot{\underline{\varepsilon}}_e + \underline{\sigma} : \dot{\underline{\varepsilon}}_v + [\underline{\sigma} : \dot{\underline{\varepsilon}}_t - \rho \frac{\partial A}{\partial \underline{\alpha}} : \dot{\underline{\alpha}}] - \rho \frac{\partial A}{\partial p} \dot{p} \geq 0 \tag{6}$$

If $\underline{\varepsilon}_e$ can be varied independently of the internal and the nonelastic deformation variables, the above inequality is satisfied for arbitrary values of the elastic strain-rate as long as the following relationships hold:

$$\underline{\sigma} = \rho \frac{\partial A}{\partial \underline{\varepsilon}_e} \tag{7}$$

$$\underline{\sigma} : \dot{\underline{\varepsilon}}_v + [\underline{\sigma} : \dot{\underline{\varepsilon}}_t - \rho \frac{\partial A}{\partial \underline{\alpha}} : \dot{\underline{\alpha}}] - \rho \frac{\partial A}{\partial p} \dot{p} \geq 0 \tag{8}$$

Since steady-flow is taken to be independent of transient processes, inequality (8) can be re-written as:

$$\underline{\sigma} : \dot{\underline{\varepsilon}}_v = \phi_v \geq 0 \tag{9}$$

where ϕ_v denotes the dissipative power per unit volume due to steady-flow, and

$$[\underline{\sigma} : \dot{\underline{\varepsilon}}_t - \rho \frac{\partial A}{\partial \underline{\alpha}} : \dot{\underline{\alpha}}] - \rho \frac{\partial A}{\partial p} \dot{p} = \phi_{tp} \geq 0 \tag{10}$$

where ϕ_{tp} denotes the dissipative power per unit volume due to the complex process of transient-flow. Equation (7) is one of the equations-of-state, while inequalities (9) and (10) represent thermodynamic restrictions on the internal processes. Inequality (9) states that the driving force associated with steady-flow is the stress tensor. On the other hand, transient deformation results from structural changes represented by $\underline{\alpha}$ and p. The tensor $\underline{\alpha}$ may be considered as a recoverable anelastic (or delayed elastic) strain associated with the elastic back-stresses generated during kinematic hardening. Analysis of experimental data suggests that the transient strain is recoverable while the steady-state strain is irrecoverable (Sinha, 1978). Consequently, the kinematic hardening variable is identified as the transient strain tensor. Isotropic hardening as represented by p is taken into account through the dependence of the complex dissipation potential on both internal variables, as described in Section 3. With the equivalence between $\underline{\alpha}$ and $\underline{\varepsilon}_t$, inequality (10) then shows that the thermodynamic force driving transient flow is the tensor $\underline{\sigma} - \rho\, \partial A/\partial\, \underline{\alpha}$, while the driving forces for the individual kinematic and isotropic hardening processes are the tensors $\rho\, \partial A/\partial\, \underline{\alpha}$ and $\rho\, \partial A/\partial p$, respectively.

Helmholtz Free Energy Potential and Thermodynamic Forces.-- The Helmholtz free energy (HFE) repesents the stored energy of the material. The existence of the free energy compatible with the second law of thermodynamics for rate-type viscoelastic and viscoplastic models has been established by Gurtin et al. (1980). For internal variable type of models, the free energy can be postulated in terms of the identified deformation mechanisms and their coupling, if any (see e.g., Lemaitre, 1987). Such postulates

are motivated by physical reasoning and experimental observations (see, e.g., Ilankamban and Krajcinovic, 1987).

In this model, the HFE consists of the instantaneous free energy defined as the strain-energy function corresponding to classical linear orthotropic elasticity. A second contribution, the anelastic free energy, is defined as the strain-energy function correponding to the elastic back-stresses. The final contribution, $A_p(p)$, takes into account the energy storage in the material resulting from internal structural changes due to isotropic hardening. The time-derivative of p is related to the equivalent transient strain-rate as shown subsequently. The analytical expression for the HFE then takes the following form:

$$\rho A = 1/2\ \underline{\varepsilon}_e : \underline{D} : \underline{\varepsilon}_e + 1/2\ A^* \underline{\alpha} :\ \underline{H} : \underline{\alpha} + \rho A_p(p) \tag{11}$$

where $\underline{D}$ is the orthotropic elastic stiffness tensor, $\underline{H}$ is a fourth-order transformation tensor for incompressible and othotropic materials (derived in Section 3), and A^* is a scalar constant which relates equivalent values of the back-stress and anelastic-strain tensors, respectively.

Direct formulation of $A_p(p)$ on the basis of physical models is complicated. The usual approach for isotropically hardening plastic and viscoplastic materials is to assume that p is a measure of the accumulated plastic strain and that isotropic hardening is a nonlinear function of p (Chaboche and Rousselier, 1983). The flux for p is obtained from a dissipation potential postulated on the basis of experimental data. In simple cases, it is possible to reconstruct an explicit expression for the associated free energy $A_p(p)$. A similar approach is adopted here as described in Section 3.

The thermodynamic forces are by definition conjugate to the state variables, as suggested by the structure of the dissipative inequality (10). Using Eq. (11), the equations-of-state take the following form:

$$\rho \frac{\partial A}{\partial \underline{\varepsilon}_e} = \underline{D} : \underline{\varepsilon}_e \equiv \underline{\sigma} \tag{12}$$

$$\rho \frac{\partial A}{\partial \underline{\alpha}} = A^* \underline{H} : \underline{\alpha} = A^* \underline{H} : \underline{\varepsilon}_t \equiv \underline{R} \tag{13}$$

$$\rho \frac{\partial A}{\partial p} = \rho \frac{\partial A_p}{\partial p} \equiv B'V \tag{14}$$

where the product $A^* \underline{H} : \underline{\alpha}$ is the the elastic back-stress tensor $\underline{R}$, B' is a nondimensional scalar drag-stress associated with isotropic hardening, and V is a stress-factor characterizing the viscous resistance of the material. Equation (12) states that the stress tensor is the thermodynamic force associated with the elastic-strain tensor, while Eq. (13) states that the back-stress tensor is the thermodynamic force associated with the anelastic-strain tensor. It should be noted that the driving force for transient-flow is the stress difference tensor $\underline{\sigma}_d \equiv \underline{\sigma} - \underline{R}$, as can be seen from Eqs. (10) and (13). Finally, Eq. (14) states that the drag-stress B'V is the thermodynamic force associated with isotropic hardening.

3. DISSIPATION POTENTIALS AND EVOLUTION EQUATIONS

Conceptual Formulation.-- The dissipation potential may be expressed as a function of either the thermodynamic fluxes or the thermodynamic forces. It is possible to derive one representation from the other via a complete Legendre transformation. A partial transformation will lead to both flux and force quantities as variables. In the model, the dissipation potential is expressed as a function of the forces and the flux quantities are obtained by imposing normality as in the classical theory of plasticity. Thermodynamic requirements are automatically satisfied if the dissipation surfaces in the hyper-space spanned by the forces are convex, nested, and contain the origin (see, e.g., Onat and Leckie, 1988).

The transient- and steady-flow components of deformation are considered to be independent. Consequently, the dissipation potential ϕ is additively decomposed into two components, one for each deformation mechanism, i.e.:

$$\phi = \phi_v + \phi_{tp} \tag{15}$$

where ϕ_v and ϕ_{tp} are the dissipation potentials for steady- and transient-flow, respectively. Normality can be applied separately to the two potentials, but not to the compound potential ϕ. In the case of steady-flow, normality is with respect to the associated force $\underline{\sigma}$, and ϕ_v is given by the Norton-Hoff type of expression for the power-law. In the case of transient-flow, the existence of a potential with the normality condition is postulated. This potential generalizes ϕ_v by replacing $\underline{\sigma}$ and V in ϕ_v by $\underline{\sigma}_d$ and B'V, respectively. It represents a complex potential, i.e., it is a function of the two thermodynamic forces $\underline{\sigma}_d$ and B'V. Given this postulate, normality is imposed with respect to each of the thermodynamic forces to obtain the corresponding flux quantities (Ziegler, 1983).

The derivation of the fluxes from the potentials is presented in the following, where for convenience engineering notation in Cartesian coordinates replaces tensor notation. Thus, for example, the strain vector is defined as $\underline{\varepsilon} = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T$, where T denotes the transpose operation.

Orthotropic Model of Steady-State Flow.-- To derive the relationship between the steady viscous strain-rate vector $\dot{\underline{\varepsilon}}_v$ (flux) and the stress vector $\underline{\sigma}$ (force), an equivalent stress measure generalized for pressure-insensitive orthotropic materials, i.e., with identical behavior in tension and compression, is defined:

$$\sigma_{eq}^2 = 3/\beta \left[a_1/3 (\sigma_{xx} - \sigma_{yy})^2 + a_2/3 (\sigma_{yy} - \sigma_{zz})^2 + a_3/3 (\sigma_{zz} - \sigma_{xx})^2 + 2a_4\sigma_{xy}^2 + 2a_5\sigma_{yz}^2 + 2a_6\sigma_{zx}^2 \right] \tag{16}$$

with β chosen to be $(a_1 + a_2)$ so that $\sigma_{eq} = \sigma_{yy}$ when the stress components are described by the vector $\underline{\sigma} = [0\ \sigma_{yy}\ 0\ 0\ 0\ 0]^T$, i.e., the y-axis is chosen to be the reference direction. Equation (16) is similar in form to that used by Hill (1950) to describe the plastic yield surface in metals possessing an orthotropic texture. Since steady-state (viscous) flow in polycrystalline ice is incompressible, it is unnecessary to consider the complex formulations of Reinicke and Ralston (1977) and Pariseau (1968) which seek to model the combined effects of incompressible ductile flow and pressure-sensitive brittle behavior with a hypothetical

"yield" surface. The equivalent stress may be expressed in compact form using matrix notation as:

$$\sigma_{eq}^2 = 3/\beta\, \underline{\sigma}^T\, \underline{G}\, \underline{\sigma} \tag{17}$$

where the stress-transformation matrix $\underline{G}$ is given by:

$$\underline{G} = \begin{bmatrix} \frac{(a_1+a_3)}{3} & \frac{-a_1}{3} & \frac{-a_3}{3} & & & \\ & \frac{(a_1+a_2)}{3} & \frac{-a_2}{3} & & \underline{0} & \\ & & \frac{(a_2+a_3)}{3} & & & \\ & & & 2a_4 & & \\ & \text{SYM.} & & & 2a_5 & \\ & & & & & 2a_6 \end{bmatrix} \tag{18}$$

If steady-state viscous flow in polycrystalline ice is taken to follow Glen's (1955) power-law, a scalar-valued dissipation potential of the Norton-Hoff type can be defined and expressed as:

$$\phi_v = \dot{\varepsilon}_o \frac{\sigma_{eq}^{N+1}}{V^N} \tag{19}$$

where N is the power-law index, $\dot{\varepsilon}_o$ is a reference strain-rate (may be set equal to unity without loss of generality), and V is a temperature-dependent factor representing the stress corresponding to the reference strain-rate in the y-direction. The temperature dependence of the stress-factor V is characterized by an Arrhenius relation (see, e.g., the review by Mellor, 1980):

$$V = V_o \exp(Q/NRT) \tag{20}$$

where T is the temperature in Kelvin, V_o is a temperature-independent constant, Q is the activation energy, and R is the universal gas constant equal to 8.31 J mol^{-1} K^{-1}. It should be noted that the Arrhenius law is generally invalid at temperatures close to the freezing point (e.g., above -10°C).

Normality between the thermodynamic flux and force requires:

$$\underline{\dot{\varepsilon}}_v = \upsilon_v \frac{\partial \phi_v}{\partial \underline{\sigma}} \tag{21}$$

where υ_v is a constant of proportionality given by (Ziegler and Wehrli, 1987):

$$\upsilon_v = \phi_v \left(\left[\frac{\partial \phi_v}{\partial \underline{\sigma}} \right]^T \underline{\sigma} \right)^{-1} \tag{22}$$

Using Eqs. (17) and (19), this constant is determined to be 1/(N+1). Noting this result, Eqs. (17), (19) and (21) can be combined to yield the desired relationship:

$$\dot{\underline{\varepsilon}}_v / \dot{\varepsilon}_o = \lambda \, \underline{S}^* \tag{23}$$

where

$$\lambda = 3/\beta \, (1/V)^N \, \sigma_{eq}^{N-1} \tag{24}$$

and

$$\underline{S}^* = \underline{G} \, \underline{\sigma} \tag{25}$$

Note that $\underline{S}^*$ is a pseudo-deviatoric stress vector for orthotropic materials. For isotropic polycrystalline ice (i.e., a_1 to a_6 = 1), $\underline{S}^*$ reduces to the conventional deviatoric-stress vector, σ_{eq} reduces to the conventional equivalent stress measure for isotropic materials, and Eq. (23) becomes the well-known three-dimensional generalization of the power-law for the creep of isotropic materials, as presented by Palmer (1967) for glacier-flow. Such a model can also be derived from isotropic tensor function theory (see, e.g., Rabotnov, 1969).

It can be verified that the steady-state viscous process satisfies the dissipative inequality (9), i.e., $\underline{\sigma}^T \dot{\underline{\varepsilon}}_v \geq 0$. Combining Eqs. (23)-(25) with the inequality yields $(\dot{\varepsilon}_o / V^N) \, \sigma_{eq}^{N+1}$, which is the dissipation function $\phi_v \geq 0$, as expected.

Evaluation of an equivalent strain or equivalent strain-rate is required, for example, to define the free energy terms in Eq. (11) and to determine whether the material is being loaded or unloaded. Using the hypothesis of energy equivalence, the relationship between the equivalent stress defined in Eq. (16) and an equivalent strain-rate measure can be established. The rate of dissipation of energy per unit volume, ϕ_v, is given by:

$$\phi_v = \underline{\sigma}^T \dot{\underline{\varepsilon}}_v \tag{26}$$

Application of the hypothesis then yields:

$$\underline{\sigma}^T \dot{\underline{\varepsilon}}_v = \sigma_{eq} \, \dot{\varepsilon}_{v,eq} \tag{27}$$

where $\dot{\varepsilon}_{v,eq}$ is the equivalent steady-state strain-rate. The steady-state strain-rate vector in Eq. (27) can be eliminated using Eqs. (23), (25) and (17) in succession to yield:

$$\dot{\varepsilon}_{v,eq} / \dot{\varepsilon}_o = (\sigma_{eq} / V)^N \tag{28}$$

Given the equivalent stress measure, Eq. (28) can be used to compute the equivalent steady-state strain-rate. Alternatively, an explicit expression can be derived by first eliminating $(\sigma_{xx} - \sigma_{yy})^2$, ..., σ_{zx}^2 in Eq. (16) through the use of Eq. (23) and then substituting the resulting expression for σ_{eq} in Eq. (28). The final expression can be epressed in compact notation as follows:

$$\dot{\varepsilon}_{v,eq}^{\ 2} = \beta/3\ \underline{\dot{\varepsilon}}_v^T\ \underline{H}\ \underline{\dot{\varepsilon}}_v \tag{29}$$

where the strain-rate transformation matrix $\underline{H}$ is given by:

$$\underline{H} = \begin{bmatrix} \frac{3(a_1+a_3)a_2^{\ 2}}{a^{*2}} & \frac{-3a_1a_2a_3}{a^{*2}} & \frac{-3a_1a_2a_3}{a^{*2}} & & & \\ & \frac{3(a_1+a_2)a_3^{\ 2}}{a^{*2}} & \frac{-3a_1a_2a_3}{a^{*2}} & & \underline{0} & \\ & & \frac{3(a_2+a_3)a_1^{\ 2}}{a^{*2}} & & & \\ & & & \frac{1}{(2a_4)} & & \\ & \text{SYM.} & & & \frac{1}{(2a_5)} & \\ & & & & & \frac{1}{(2a_6)} \end{bmatrix} \tag{30}$$

with $a^* = a_2a_3 + a_3a_1 + a_1a_2$. Equation (29) reduces to the conventinal equivalent strain-rate measure for isotropic materials if the orthotropic material parameters a_1 to $a_6 = 1$. Moreover, when loading is in the reference direction, $\dot{\varepsilon}_{v,eq} = \dot{\varepsilon}_{v,yy}$, and Eq. (28) becomes the power-law for uniaxial loading in the reference direction.

It should be remarked that $\underline{H}$ is not the inverse of $\underline{G}$. Furthermore, these matrices represent completely symmetric fourth rank tensors, and the relationship between them can be derived in the following steps: (i) replace the steady-state strain-rate vector in Eq. (29) by stress quantities using Eqs. (23)-(25); (ii) compare the resulting expression with Eq. (28); and (iii) carry out the appropriate cancellations and, finally, rewrite the equivalent stress in the form of Eq. (17). The final result is $\underline{\sigma}^T(\underline{G}^T\ \underline{H}\ \underline{G})\ \underline{\sigma} = \underline{\sigma}^T\ \underline{G}\ \underline{\sigma}$, thus yielding:

$$\underline{G}^T\ \underline{H}\ \underline{G} = \underline{G} \tag{31}$$

Equation (31) can also be directly verified using Eqs. (18) and (30). The quantity $\underline{G}\ \underline{H}$ has the property that $\underline{G}\ \underline{H}\ [\underline{\varepsilon}_v] = \underline{\varepsilon}_v$, where the tensor form of $\underline{\varepsilon}_v$ is traceless. $\underline{H}$ may be considered as a semi-inverse of $\underline{G}$ when the operand is deviatoric, which again can be verified by direct substitution.

Orthotropic Model of Transient Flow.-- The model development assumes that the transient deformation is incompressible. Although this may not be strictly true, Sinha (1987) argues that transient flow does not change the volume appreciably and that the assumption is valid for fresh-water polycrystalline ice. A second assumption is that material orthotropy can be described by the same set of parameters, i.e., a_1 through a_6. Similar assumptions are made in creep models for metals (e.g., Hart, 1976; Miller, 1976). Consequently, equivalent measures for the thermodynamic forces and their conjugate variables can be defined with respect to the same stress- and strain-rate transformation matrices used in Eqs. (17) and (29).

The relationship between the thermodynamic fluxes and their conjugate variables, i.e., between $\underline{\dot{\varepsilon}}_t$ and $\underline{\sigma}_d$, and between $\dot{p}$ and B'V, are derived from the scalar-valued complex potential of dissipation expressed in terms of $\underline{\sigma}_d$ and B'V:

$$\phi_{tp} = \dot{\varepsilon}_o \frac{\sigma_{d,eq}^{N+1}}{(B'V)^N} \tag{32}$$

$$\underline{\dot{\varepsilon}}_t = \upsilon_{tp} \frac{\partial \phi_{tp}}{\partial \underline{\sigma}_d} \tag{33}$$

$$\dot{p} = -\upsilon_{tp} \frac{\partial \phi_{tp}}{\partial (B'V)} \tag{34}$$

where υ_{tp} is the constant of proportionality given by (Ziegler and Wehrli, 1987):

$$\upsilon_{tp} = \phi_{tp} \left(\left[\frac{\partial \phi_{tp}}{\partial \underline{\sigma}_d}\right]^T \underline{\sigma}_d + \left[\frac{\partial \phi_{tp}}{\partial (B'V)}\right] B'V \right)^{-1} \tag{35}$$

and $\sigma_{d,eq}^2 = 3/\beta\ \underline{\sigma}_d^T\ \underline{G}\ \underline{\sigma}_d$ as in Eq. (17). From Eqs. (35) and (32) υ_{tp} is determined to be unity. Also, differentiation in Eq. (33) yields:

$$\underline{\dot{\varepsilon}}_t / \dot{\varepsilon}_o = \lambda_d\ \underline{S}_d^* \tag{36}$$

$$\lambda_d = 3/\beta\ [1/(BV)]^N\ \sigma_{d,eq}^{N-1} \tag{37}$$

and

$$\underline{S}_d^* = \underline{G}\ \underline{\sigma}_d = \underline{G}\ \underline{\sigma} - \underline{G}\ \underline{R} = \underline{S}^* - \underline{S}_R^* \tag{38}$$

where B in Eq. (37) equals $B'/(N+1)^{1/N}$. Equations (36)-(38) are the counterparts of Eqs. (23)-(25) for steady-flow. The transient strain-rate reverses direction when the applied loading $\underline{S}^*$ is less than the back-stress measure $\underline{S}_R^*$. Note that the equation for $\underline{S}_R^*$ is already given by Eq. (13). By premultiplying Eq. (13) with $\underline{G}$ and using the identity $\underline{G}\ \underline{H}\ \underline{\varepsilon}_t = \underline{\varepsilon}_t$ (Eq. (31)), which is valid for incompressible flow, i.e., $\varepsilon_{t,xx} + \varepsilon_{t,yy} + \varepsilon_{t,zz} = 0$, an alternative expression can be obtained as given below:

$$\underline{S}_R^* = \underline{G}\,\underline{R} = \beta/3\ AE\ \underline{\varepsilon}_t \tag{39}$$

The parameter A^* of Eq. (11) has been replaced by $(\beta/3)$ AE, where $\beta/3$ is the scalar defining the equivalent strain-rate (see Eq. (29)). It follows that AE is the scalar rigidity-constant linearly relating equivalent values of the back-stress and transient strain, and E is the Young's modulus of polycrystalline ice in the reference direction.

The dissipative process associated with transient-strain satisfies the dissipative inequality (10) automatically, since $[\underline{\sigma} - \rho\ \partial A/\partial \underline{\varepsilon}_t]^T\ \dot{\underline{\varepsilon}}_t - [\rho\ \partial A_p/\partial p]\ \dot{p}$ in the inequality is simply ϕ_{tp}, as can be verified by noting that in Eq. (35) υ_{tp} equals one and the terms within the parenthesis are precisely those in inequality (10). Since $\phi_{tp} \geq 0$, as indicated by Eq. (32), thermodynamic requirements are satisfied.

The evolution equation for the isotropic hardening variable p follows from Eqs. (32) and (34):

$$\dot{p} / \dot{\varepsilon}_o = k \left[\frac{\sigma_{d,eq}}{BV}\right]^{N+1} \tag{40}$$

where for ξ defined as (N+1)/N, k equals $N/(N+1)^{\xi}$. This equation completes the model formulation if an analytical expression can be found for the free energy $A_p(p)$, since the equation-of-state, Eq. (14), will then give an analytical relation between B'V (force) and P (conjugate variable). Instead, Shyam Sunder and Wu (1988) have independently postulated the following rule for isotropic hardening:

$$\dot{B} = (\bar{H}\, E/\sigma_{d,eq})\ \dot{\varepsilon}_{t,eq} \tag{41}$$

where $\bar{H}$ is a temperature-independent constant and $\dot{\varepsilon}_{t,eq}$ is the equivalent transient strain-rate specified by $\underline{H}$ as in Eq. (29). The initial value of B, denoted by B_o, represents initial hardening introduced by pre-straining or the presence of residual-stress in a previously undeformed material. The value of B increases during loading (where loading is defined as an increase in value of the equivalent transient-strain). If unloading occurs, i.e., the value of the equivalent transient-strain decreases, the sign of Eq. (41) is reversed and the dimensionless drag-stress decreases.

This formulation for isotropic hardening is motivated by the work of Ashby and Duval (1985) who constructed master-curves relating dimensionless measures of strain-rate, strain and time using Jacka's (1984) comprehensive creep data on polycrystalline ice. These master-curves are independent of temperature and applied stress. Shyam Sunder and Wu (1988) have shown that the isotropic hardening rule given by Eq. (41) satisfies the dimensional requirements identified by Ashby and Duval (1985) for both constant-stress and constant strain-rate responses. Model predictions using this rule have been validated against several independent sets of uniaxial creep and recovery data.

A mathematical expression for the free energy $A_p(p)$ corresponding to the isotropic hardening rule of Eq. (41) can also be reconstructed. However, it is necessary to establish first the relationship between $\dot{p}$ and $\dot{\varepsilon}_{t,eq}$. Noting that $\underline{G}^T\,\underline{H}\,\underline{G} = \underline{G}$ (Eq. (31)), it can be shown from Eqs. (36)-(38) that:

$$\dot{\varepsilon}_{t,eq} / \dot{\varepsilon}_o = [\sigma_{d,eq}/(BV)]^N \tag{42}$$

Substituting Eq. (42) in Eq. (40) yields:

$$\dot{p} / \dot{\varepsilon}_o = k (\dot{\varepsilon}_{t,eq}/ \dot{\varepsilon}_o)^{\xi} \tag{43}$$

which establishes that p is a measure of the accumulated transient-strain. A relationship involving only B, $\dot{B}$ and $\dot{p}$ is then obtained by eliminating $\sigma_{d,eq}$ and $\dot{\varepsilon}_{t,eq}$ from Eqs. (41)-(43), i.e.:

$$(\dot{p}/\dot{\varepsilon}_o)^{m} = k^m BV/(\bar{H} E) (\dot{B}/ \dot{\varepsilon}_o) \tag{44}$$

where m = (N-1)/(N+1). Equation (44) is in effect the evolution equation for p. Integrating Eq. (44) by separation of variables yields a relation between B and $\dot{p}$, i.e.:

$$B^2 = B_o^2 + 2 (\bar{H} E/V) k^{-m} \dot{\varepsilon}_o \int (\dot{p} / \dot{\varepsilon}_o)^{m} dt \tag{45}$$

The desired expression for $A_p(p)$ follows by integrating the thermodynamic force B'V with respect to the flux p:

$$\rho A_p = \int B'V \, dp = \int \{ [B_o^2 + 2 (\bar{H} E/V) k^{-m} \dot{\varepsilon}_o \int (\dot{p} / \dot{\varepsilon}_o)^{m} dt]^{1/2} \} V(N+1)^{1/N} dp \tag{46}$$

Summary of Model Equations.-- Using tensor notation, the governing equations of the model are summarized in the following under three categories:

(a) Strain-Rate Decomposition

$$\dot{\underline{\varepsilon}} = \dot{\underline{\varepsilon}}_e + \dot{\underline{\varepsilon}}_t + \dot{\underline{\varepsilon}}_v \tag{47}$$

(b) Equations-of-State (In Rate Form)

$$\dot{\underline{\sigma}} = \underline{D} : \dot{\underline{\varepsilon}}_e = \underline{D} : (\dot{\underline{\varepsilon}} - \dot{\underline{\varepsilon}}_t - \dot{\underline{\varepsilon}}_v) \tag{48}$$

$$\dot{\underline{\sigma}}_d = \dot{\underline{\sigma}} - \dot{\underline{R}} \tag{49}$$

where

$$\dot{\underline{R}} = (\beta/3) AE \, \underline{H} : \dot{\underline{\varepsilon}}_t \quad \text{or} \quad \dot{\underline{S}}_R^* = (\beta/3) AE \, \dot{\underline{\varepsilon}}_t \tag{50}$$

(c) Evolution Equations

$$\dot{\underline{\varepsilon}}_v / \dot{\varepsilon}_o = \lambda \underline{S}^* \tag{51}$$

$$\dot{\underline{\varepsilon}}_t / \dot{\varepsilon}_o = \lambda_d \underline{S}_d^* = \lambda_d (\underline{S}^* - \underline{S}_R^*) \tag{52}$$

$$\dot{B} = (\bar{H} E / \sigma_{d,eq}) \dot{\varepsilon}_{t,eq} \tag{53}$$

The evolution equations are highly nonlinear and are coupled with the equations-of-state. Numerical integration is required for the solution of the governing model equations, except for the special case of uniaxial constant-stress loading and no isotropic hardening (Shyam Sunder and Wu, 1988). In problems involving variable loading histories, the high stiffness of these equations pose severe constraints on the development of a finite element solution. Shyam Sunder, Wu and Chen (1988) have used a Newton-Raphson iteration technique combined with the α-method of time integration to ensure and accelerate convergence. The discretized system of finite element equations are solved with an incremental-iterative method in which the incremental tangent- stiffness matrix is updated at each iteration using the BFGS technique (see, e.g., Matthies and Strang, 1979).

4. ESTIMATION OF MODEL PARAMETERS

The model parameters N, E, V_o, A, B_o and $\bar{H}$ have been estimated from uniaxial compressive creep tests on polycrystalline ice (Shyam Sunder and Wu, 1988). The values of the six parameters, corresponding to the creep data of Jacka (1984) and Sinha (1978), are listed in Table 1. Jacka's data are obtained from constant-stress creep tests on isotropic polycrystalline ice with a mean grain size of 1.7 mm, while Sinha's data are obtained from similar tests on transversely isotropic columnar-grained (S-2) ice with a mean grain size of 3 mm. The different parameter values reflect the different types of ice that were tested. The value of the activation energy Q has been estimated to be about 67 KJ mol^{-1} (Gold, 1973; Sinha, 1978).

For pure flow at stresses of engineering interest, the value of the power-law constant N is approximately three. In a paper reviewing the constants used in Glen's power law for polycrystalline glacier ice, Hooke (1981) concludes that in the absence of experimental evidence to the contrary, a value of three for N is reasonable, irrespective of the "structural state", e.g., texture. The effect of the structural state is accounted for by changing the "viscosity". This is the approach adopted here, in which N is taken to be three and the initial texture or material anisotropy is accounted for by defining appropriate measures of equivalent stress and strain-rate. Equations (24) and (36) show that the "effective viscosity" in different directions during steady-state and transient-flow is determined by the orthotropic parameters a_1 through a_6. As in the case of N, the parameters V_o, A, B_o and $\bar{H}$ are determined from uniaxial tests on polycrystalline ice in the reference direction and are constants.

The orthotropic parameters a_1 to a_6 may be estimated from experimental data under steady-flow conditions. Noting that a_1 can be set to unity without loss of generality, the remaining constants may be obtained from five uniaxial (compression) constant-stress creep or constant strain-rate loading as shown below.

Consider a Cartesian frame defined on an orthotropic ice sheet. The x-axis is taken to be normal to the ice sheet which is defined by the y-z plane. The c-axes of the ice crystals are assumed to lie in the y-z plane and are aligned in the y-direction. The tests are conducted in the three orthogonal directions y, x, and z respectively, and along the three 45° axes on the y-z, x-y, and z-x planes respectively. Let β_1 to β_5 represent the experimentally determined ratios of the maximum stresses for the latter five tests, respectively, to the maximum stress in the reference y-direction for tests conducted at the same constant strain-rate. In the case of creep tests, the β's represent inverse ratios of the corresponding minimum strain-rates raised to the power of 1/N. The orthotropic model parameters are explicitly related to the β's as shown in the following.

Applying Eqs. (23) and (16) to each of the unaxial tests, expressing the equations in terms of the β's, and solving the resulting equations yields the following equations for the parameters a_2 to a_6 (see Ting and Shyam Sunder, 1985, for additional details):

$$a_2 = - \frac{\beta_1^n - \beta_2^n(1-\beta_1^n)}{\beta_1^n - \beta_2^n(1+\beta_1^n)} \tag{54}$$

$$a_3 = - \frac{\beta_1^n + \beta_2^n(1-\beta_1^n)}{\beta_1^n - \beta_2^n(1+\beta_1^n)} \tag{55}$$

$$a_4 = \beta/6 \, [4\beta_4^{-n} - \beta_2^{-n}] \tag{56}$$

$$a_5 = \beta/6 \, [4\beta_3^{-n} - \beta_1^{-n}] \tag{57}$$

$$a_6 = \beta/6 \, [4\beta_5^{-n} - 1] \tag{58}$$

where $n = 2N/(N+1)$. Note that Mohr's circle transformations are necessary for the stresses and steady-state strain-rates when deriving Eqs. (56)-(58). These five equations provide a physical interpretation of the orthotropic model parameters.

For transversely-isotropic ice, i.e., isotropy in the y-z plane, $\beta_2 = \beta_3 = 1$ and $\beta_4 = \beta_5$. As a result, $a_1 = a_3 = 1$, $a_4 = a_6$, the parameters a_2 and a_5 are functions of only β_1, while a_4 depends on both β_1 and β_4. Only two uniaxial tests are required to determine β_1 and β_4: one in the x-direction and one along the 45° axis on the x-y or z-x planes.

The necessary experimental data on orthotropic or transversely-isotropic fresh-water polycrystalline ice under pure flow (ductile) conditions for determining these parameters is not readily available. However, estimates for the parameter ranges in the case of β_1 - β_3 may be obtained from tests conducted under non-ideal conditions. For example, in the case of freshwater polycrystalline (S-2) ice β_1, the ratio of out-of-plane to in-plane maximum stress, is around two at strain-rates involving some brittle behavior (Carter and Michel, 1971). The values of β_1 to β_3 have been reported for sea ice (see, e.g., Richter-Menge, 1987; Sinha, 1983; Vittoratos, 1979; Wang, 1979). Typical ranges are 2 - 5 for β_1, 0.50 - 0.95 for β_2, and 0.25 - 0.60 for β_3.

Plane-strain compression tests on transversely isotropic ice such as those conducted by Frederking

(1977) may also be considered for determining β_1. For his type A tests, strains in the z-direction are constrained to zero and stresses are applied in the y-direction. The ratio γ_z of the plane-strain maximum stress to the unconfined maximum stress at the same strain-rate can be derived following the procedure used for Eqs. (54) - (58), i.e.:

$$\gamma_z = \left[\frac{4\beta_1^{2n}}{4\beta_1^n - 1}\right]^{1/n} \tag{59}$$

This equation predicts γ_z to vary between 2.1 to 5.1 for values of β_1 ranging from 2 to 5, and N = 3. Frederking's experiments, although involving brittle behavior, yield γ_z values close to two at high strain-rates and to five at low strain-rates. In the type B tests, strains in the x-direction are constrained to zero while stresses are again applied in the y-direction. In this case, the stress ratio γ_x can be expressed as:

$$\gamma_x = \left[1 + \frac{1}{4\beta_1^n - 1}\right]^{1/n} \tag{60}$$

Since β_1 is generally greater than one, γ_x will be less than approximately 1.2 for N = 3. The equation predicts γ_x values from 1.01 to 1.06 for typical values of β_1. These results show that for transversely-isotropic ice the type A test is almost a direct measure of β_1, while the type B test is insensitive to typical variations in β_1 and consequently is not a robust test for determining β_1.

5. CONCLUSIONS

A multiaxial differential model of flow in orthotropic polycrystalline ice is presented in this paper. The derivation of the constitutive equations is based on thermodynamics with internal state variables. These internal variables are taken to represent structural changes on the microscale associated with isotropic and kinematic hardening. Use of the dissipative inequality identifies the stress tensor as the force driving steady-state viscous flow and a stress-difference tensor, equal to the stress tensor minus the elastic back-stress tensor associated with kinematic hardening, as the force driving transient-flow in the material.

The evolution equations for the steady-state and transient deformation tensors are derived from independent potential functions characterizing the dissipation process by imposing normality with respect to the appropriate conjugate forces. Dissipation during transient-flow is accompanied by energy storage resulting from internal structural changes. The kinematic hardening variable is identified as the transient-strain tensor and is linearly related to the back-stress. The flux quantity corresponding to isotropic hardening is obtained from the same complex potential used in deriving the evolution of the transient deformation tensor by imposing normality with respect to a scalar drag-stress. It is shown that this flux is nonlinearly related to the equivalent transient strain-rate.

The model uses five parameters to represent the orthotropic texture of ice during incompressible flow. These parameters may be determined from either constant-stress creep or constant strain-rate tests

under steady-flow conditions. Conventional uniaxial compression testing is sufficient to determine these parameters. In the case of transversely-isotropic ice, two of the three independent parameters may also be determined from plane-strain compression tests, i.e., the type A tests of Frederking (1977).

The flow model presented in this paper serves as a foundation for a rate- and pressure-sensitive damage model that can describe the effects of distributed microcracking in polycrystalline ice. Such a model based on defining a damage tensor as an additional internal variable is presented by Wu and Shyam Sunder (1989).

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Table 1. Parameters for Uniaxial Constitutive Model (From Shyam Sunder and Wu, 1988).

Parameter	Value Determined from Data of	
	Jacka (1984)	Sinha(1978)
E	9.5 GPa	9.5 GPa
N	3	3
V_0	6.59 KPa	6.59 KPa
A	0.021	0.495
B_o	0.286	0.087
H	0.020	0.454